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Radiation reaction, renormalization and conservation laws in six-dimensional classical electrodynamics

Yu Yaremko

Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine,
1 Svientsitskii St., 79011 Lviv, Ukraine

E-mail: yar@icmp.lviv.ua

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Abstract

A self-action problem for a point-like charged particle arbitrarily moving in flat spacetime of six dimensions is considered. A consistent regularization procedure is proposed which relies on energy–momentum and angular momentum balance equations. The structure of the angular momentum tensor carried by the retarded 'Liénard–Wiechert' field testifies that a point-like source in six dimensions possesses an internal angular momentum. Its magnitude is proportional to the square of acceleration. It is the so-called *rigid* relativistic particle; its motion is determined by the higher derivative Lagrangian depending on the curvature of the worldline. It is shown that the action functional contains, apart from the usual 'bare' mass, an additional renormalization constant which corresponds to the magnitude of the 'bare' internal angular momentum of the particle.

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1. Introduction

Recently [1, 2], there has been considerable interest in the renormalization procedure in classical electrodynamics of a point particle moving in flat spacetime of arbitrary dimensions. The main task is to derive the analogue of the well-known Lorentz–Dirac equation [3]. The Lorentz–Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. (For a modern review see [4–6].) In an earlier paper [7] the Lorentz–Dirac equation in six dimensions is obtained via the consideration of energy–momentum conservation.

All the authors [1, 2, 7] deal with an obvious generalization of the standard variational principle used in four dimensions

$$I = I_{\text{particle}} + I_{\text{int}} + I_{\text{field}} \tag{1.1}$$

with

$$I_{\text{field}} = -\frac{1}{4\Omega_{D-2}}\int \mathrm{d}^D y\, F^{\mu\nu}F_{\mu\nu} \qquad I_{\text{particle}} = -m\int \mathrm{d}\tau\sqrt{-\dot{z}^2} \tag{1.2}$$

and the interaction term given by

$$I_{\text{int}} = e\int \mathrm{d}\tau\, A_\mu \dot{z}^\mu. \tag{1.3}$$

By Ω_{D-2} the area of a $(D-2)$-dimensional sphere of unit radius is denoted as

$$\Omega_{D-2} = 2\frac{\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{1.4}$$

The functions $z^\mu(\tau)$ give the particle's coordinates as a function of proper time τ; $\dot{z}^\mu(\tau) = \mathrm{d}z^\mu(\tau)/\mathrm{d}\tau$.

Strictly speaking, the action integral (1.1) may be used to derive trajectories of the test particles, when the field is given *a priori*. It may also be used to derive D-dimensional Maxwell equations, if the particle trajectories are given *a priori*. Simultaneous variation with respect to both field and particle variables is incompatible since the Lorentz force will always be ill defined in the immediate vicinity of the particle's world line.

The elimination of the divergent self-energy of a point charge is the key to the problem. In four-dimensional spacetime one usually assumes that the parameter m involving in I_{particle} is the unphysical *bare* mass. It absorbs the inevitable infinity within the renormalization procedure and becomes the *observable* rest mass of the particle. In D dimensions the Coulomb potential of a charge scales as $|\mathbf{x}|^{3-D}$ [8]. Inevitable infinities arising in higher dimensional electrodynamics are stronger than in four dimensions.

Therefore, in even dimensions higher than four divergences cannot be removed by the renormalization of mass included in the initial action integral (1.1) [1, 2, 7]. To make classical electrodynamics in six dimensions a renormalizable theory, in [7] the six-dimensional analogue of the relativistic particle with rigidity [9–11] is substituted for the structureless point charge whose action term is proportional to worldline length. The corresponding Lagrangian involves, apart from usual 'bare mass', an additional regularization constant which absorbs one extra divergent term. In [2] the procedure of regularization in any dimension is elaborated. It allows removal of the infinities coming from the particle's self-action by introducing new counterterms in the particle action.

In the present paper the problem of renormalizability will be reformulated within the problem of Poincaré invariance of a closed particle plus field system. We calculate the energy–momentum and angular momentum of the retarded electromagnetic field generated by a point-like charge in six dimensions. As it is in four dimensions [12, 13], the energy-momentum contains two quite different terms [7]: (i) the bound part which is permanently 'attached' to the charge and is carried along with it; (ii) the radiation part detaches itself from the charge and leads an independent existence. The former is divergent while the latter is finite. The radiative momentum is accumulated with time while the bound one depends on the state of the particle's motion at the instant of observation only. Hence, a charged particle cannot be separated from its bound electromagnetic 'cloud', so that the six-momentum of the particle is the sum of the mechanical momentum and the electromagnetic bound energy–momentum.

Conserved quantities place stringent requirements on the dynamics of the system. They demand that the change in *radiative* energy–momentum and angular momentum should be balanced by a corresponding change in the *already renormalized* momentum and angular momentum of the particle. In four dimensions the energy–momentum balance equation gives the relativistic generalization of Newton's second law

$$\dot{p}^\mu_{\text{part}} = -\tfrac{2}{3}e^2a^2u^\mu + F^\mu_{\text{ext}} \tag{1.5}$$

where loss of energy due to radiation is taken into account (see [14]). The angular momentum balance equation explains how the four-momentum of the charged particle depends on its velocity and acceleration:

$$p^{\mu}_{\text{part}} = mu^{\mu} - \tfrac{2}{3}e^2 a^{\mu} \tag{1.6}$$

where m is the already renormalized rest mass. So, a careful analysis with the use of a regularization procedure compatible with the Poincaré symmetry leads to the Lorentz–Dirac equation in the four-dimensional case.

Similarly, in six dimensions the energy–momentum balance equations give a six-dimensional analogue of the Larmor relativistic rate of radiated energy–momentum [7]. Fifteen angular momentum balance equations will explain how the six-momentum and angular momentum of the particle depend on worldline characteristics such as six-velocity, six-acceleration, etc. Does a consistent classical electrodynamics in a spacetime of six dimensions lead inevitably to the rigid particle? If so, we should arrive at the specific angular momentum which depends on the particle's acceleration (see [9–11]).

2. General setting

Let $\mathbb{M}_6$ be six-dimensional Minkowski space with coordinates y^{μ} and metric tensor $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1, 1)$. We use the natural system of units with the velocity of light $c = 1$. Summation over repeated indices is understood throughout the paper; Greek indices run from 0 to 5, and Latin indices from 1 to 5.

We consider an electromagnetic field $F_{\alpha\beta}$ produced by a point-like particle of charge e. The particle moves in flat spacetime $\mathbb{M}_6$ on an arbitrary worldline

$$\zeta : \mathbb{R} \to \mathbb{M}_6 \qquad u \mapsto (z^{\mu}(u)) \tag{2.1}$$

where u is proper time. The Maxwell field equation are

$$F^{\alpha\beta}{}_{,\beta} = \frac{8\pi^2}{3} j^{\alpha} \tag{2.2}$$

where current density j^{α} is given by

$$j^{\alpha} = e \int \mathrm{d}\tau \, u^{\alpha}(u) \delta(y - z(u)). \tag{2.3}$$

$u^{\alpha}(u)$ denotes the (normalized) six-velocity vector $\mathrm{d}z^{\alpha}(u)/\mathrm{d}u$ and the factor $8\pi^2/3$ is the area of a four-dimensional unit sphere embedded in $\mathbb{M}_6$ (see equation (1.4) for $D = 6$).

We express the electromagnetic field in terms of a vector potential, $\hat{F} = \mathrm{d}\hat{A}$. In the Lorentz gauge $A^{\alpha}{}_{,\alpha} = 0$ the Maxwell field equations become

$$\Box A^{\alpha}(y) = -\frac{8\pi^2}{3} j^{\alpha}(y) \tag{2.4}$$

where $\Box := \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}$ is the wave operator. Using the retarded Green function [1, equation (3.4)] associated with the d'Alembert operator $\Box$ and the charge-current density vector (2.3) we construct the retarded Liénard–Wiechert potential in six dimensions:

$$A_{\mu}(y) = e \int \mathrm{d}u \, u_{\mu}(u) \left(-\frac{1}{2\pi R} \frac{\mathrm{d}}{\mathrm{d}R} \frac{\delta(T - R)}{R} \right). \tag{2.5}$$

Here $T := y^0 - z^0(u)$ and $R := |\mathbf{y} - \mathbf{z}(u)|$.

We suppose that the dynamics of our composite particle plus field system is governed by the conservation laws which arise from the invariance of the closed system under time and

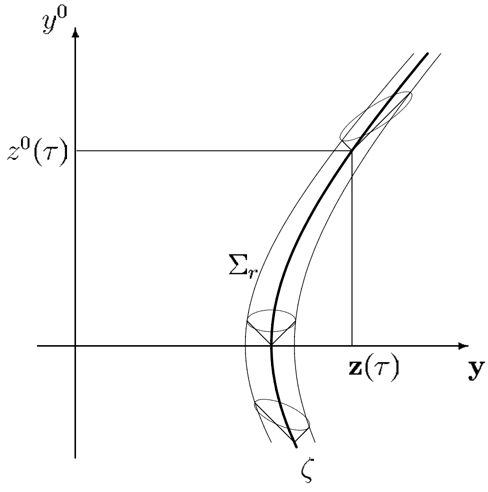

Figure 1. Integration region considered in the evaluation of the bound and emitted conserved quantities produced by all points of the worldline up to the end point $(z^0(\tau), \mathbf{z}(\tau))$. Retarded spheres $S(z(u), r)$, $u \in]-\infty, \tau]$, of constant radii r constitute a thin world tube Σ_r enclosing the worldline ζ. The sphere $S(z(u), r)$ is the intersection of the future light cone with vertex at point $z^\mu(u) \in \zeta$ and r-shifted hyperplane $\Sigma(z(u), r)$ which is orthogonal to six-velocity $u^\mu(u)$.

space translations as well as space and mixed spacetime rotations. The components of the momentum six-vector carried by the electromagnetic field are [7]

$$p^\nu_{\text{em}}(\tau) = P \int_\Sigma \mathrm{d}\sigma_\mu \, T^{\mu\nu} \tag{2.6}$$

where $\mathrm{d}\sigma_\mu$ is the vectorial surface element on an arbitrary space-like hypersurface Σ. The components of the electromagnetic field's stress–energy tensor

$$\frac{8\pi^2}{3} T^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda - 1/4 \eta^{\mu\nu} F^{\kappa\lambda} F_{\kappa\lambda} \tag{2.7}$$

have singularities on a particle trajectory (2.1). In equation (2.6) the capital letter P denotes the principal value of the singular integral, defined by removing from Σ an ε-sphere around the particle and then passing to the limit $\varepsilon \to 0$.

The angular momentum tensor of the electromagnetic field is written as [4]

$$M^{\mu\nu}_{\text{em}}(\tau) = P \int_\Sigma \mathrm{d}\sigma_\alpha (y^\mu T^{\alpha\nu} - y^\nu T^{\alpha\mu}). \tag{2.8}$$

Following [7], we calculate the energy–momentum (2.6) and angular momentum (2.8) which flow across a world tube of constant radius r enclosing the worldline ζ. This integration hypersurface, say Σ_r, is a disjoint union of (retarded) spheres of constant radii r centred on a worldline of the particle (see figure 1). The sphere $S(z(u), r)$ is the intersection of the future light cone generated by null rays emanating from $z(u) \in \zeta$ in all possible directions

$$C(z(u)) = \left\{ y \in \mathbb{M}_6 : (y^0 - z^0(u))^2 = \sum_i (y^i - z^i(u))^2, \, y^0 - z^0(u) > 0 \right\} \tag{2.9}$$

and the tilted hyperplane

$$\Sigma(z(u), r) = \{ y \in \mathbb{M}_6 : u_\alpha(u)(y^\alpha - z^\alpha(u) - u^\alpha(u) r) = 0 \}. \tag{2.10}$$

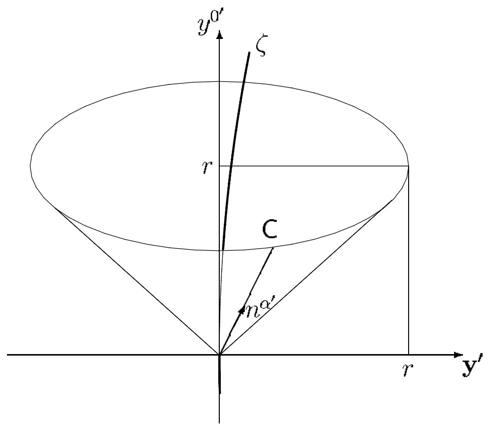

Figure 2. In MCLF the retarded distance is the distance between any point on the spherical light front $S(0,r) = \{y \in \mathbb{M}_6 : (y^{0'})^2 = \sum_i (y^{i'})^2, y^{0'} = r > 0\}$ and the particle. The charge is placed at the coordinate origin; it is momentarily at rest. The point $\mathsf{C} \in S(0,r)$ is linked to the coordinate origin by a null ray characterized by the angles ϑ^A specifying its direction on the cone. The null vector $\mathbf{n}$ with components $n^{\alpha'} = y^{\alpha'}/r$ defines this direction.

3. Coordinate system

An appropriate coordinate system is very important for the integration. We use an obvious generalization of *a coordinate system centred on an accelerated worldline* [5, 16]. The set of curvilinear coordinates for flat spacetime $\mathbb{M}_6$ involves the *retarded time u* and the *retarded distance r* introduced in the previous section (see equations (2.9) and (2.10), respectively). To understand the situation more thoroughly, we pass to the particle's momentarily comoving Lorentz frame (MCLF) where the particle is momentarily at rest at the retarded time u. The retarded distance r is the distance between an observer event $\mathsf{C} \in \Sigma_r$ and the particle, as measured at u in the MCLF (see figure 2). Points on the sphere $S(0,r) \subset \Sigma_r$ are distinguished by four spherical angles $(\phi, \vartheta_1, \vartheta_2, \vartheta_3)$:

$$
\begin{aligned}
y^{0'} = r \qquad & y^{1'} = r\cos\phi\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3 \\
& y^{2'} = r\sin\phi\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3 \\
& y^{3'} = r\cos\vartheta_1\sin\vartheta_2\sin\vartheta_3 \\
& y^{4'} = r\cos\vartheta_2\sin\vartheta_3 \\
& y^{5'} = r\cos\vartheta_3 .
\end{aligned}
\tag{3.1}
$$

In the laboratory frame the points on this sphere have the following coordinates,

$$
\begin{aligned}
y^\alpha &= z^\alpha(u) + r\Lambda^\alpha{}_{\alpha'}(u) n^{\alpha'} \\
&= z^\alpha(u) + r k^\alpha
\end{aligned}
\tag{3.2}
$$

where $n^{\alpha'} = y^{\alpha'}/r$.

We see that flat spacetime $\mathbb{M}_6$ becomes a disjoint union of world tubes $\Sigma_r, r > 0$, enclosing the particle trajectory (2.1). A world tube is a disjoint union of (retarded) spheres of constant radii r centred on a world line of the particle. Points on a sphere are distinguished by four spherical polar angles.

4. Electromagnetic potential and electromagnetic field in six dimensions

In even spacetime dimensions the Green function associated with the d'Alembert operator is localized on the light cone [1, 2]. Having integrated (2.5) we obtain the potential

$$\begin{aligned} A_\mu &= \frac{e}{2\pi}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u_\mu(u)}{r}\right) \\ &= \frac{e}{2\pi}\left[\frac{a_\mu(u)}{r^2} + \frac{u_\mu(u)}{r^3}(1+ra_k)\right] \end{aligned} \tag{4.1}$$

where $a_k = a_\alpha k^\alpha$ is the component of the acceleration $a_\alpha = \mathrm{d}u_\alpha/\mathrm{d}u$ in the direction of k^α. It is understood that in equation (4.1), all worldline quantities (such as u_μ and a_μ) are to be evaluated at the retarded time u.

The potential (4.1) differs from that of [7, equation (13)] just by an overall coefficient $e/2\pi$.

The direct particle field [17] is defined in terms of this potential by $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$. Having used the differentiation rule [7, equations (2) and (3)]

$$\frac{\partial u}{\partial y^\mu} = -k_\mu \qquad \frac{\partial r}{\partial y^\mu} = -u_\mu + (1+ra_k)k_\mu \tag{4.2}$$

we obtain

$$F = \frac{e}{2\pi}\left(\frac{u\wedge a}{r^3} + V\wedge k\right) \tag{4.3}$$

where

$$V_\mu = \frac{3u_\mu}{r^4} + \frac{3(a_\mu + 2u_\mu a_k)}{r^3} + \frac{\dot{a}_\mu + u_\mu\dot{a}_k + 3a_\mu a_k + 3u_\mu a_k^2}{r^2}. \tag{4.4}$$

The overdot means the derivative with respect to retarded time u. Liénard–Wiechert field (4.3) coincides with the field obtained in [7, equation (14)] where the 'mostly minus' metric signature should be replaced by the 'mostly plus' one.

5. Energy–momentum of the retarded Liénard–Wiechert field in six dimensions

It is straightforward to substitute the components (4.3) into equation (2.7) to calculate the stress–energy tensor of the electromagnetic field. Following [7], we present $T^{\alpha\beta}$ as a sum of radiative and bound components

$$T^{\alpha\beta} = T^{\alpha\beta}_{\mathrm{rad}} + T^{\alpha\beta}_{\mathrm{bnd}}. \tag{5.1}$$

The radiative part scales as r^{-4}:

$$\frac{8\pi^2}{3}T^{\alpha\beta}_{\mathrm{rad}} = \frac{e^2}{4\pi^2}\frac{k^\alpha k^\beta}{r^4}V^\mu_{(-2)}V^{(-2)}_\mu \tag{5.2}$$

where the components $V^\mu_{(-2)}$ of six-vector $V_{(-2)}$ are defined by equation (4.4). The others $T_{(-\kappa)}$ constitute the bound part of the Maxwell energy–momentum tensor density:

$$T^{\alpha\beta}_{\mathrm{bnd}} = T_{(-8)} + T_{(-7)} + T_{(-6)} + T_{(-5)}. \tag{5.3}$$

(Each term has been labelled according to its dependence on the distance r.)

According to [7], the outward-directed surface element $\mathrm{d}\sigma_\mu$ of a five-cylinder $r = \mathrm{const}$ in $\mathbb{M}_6$ is

$$d\sigma_\mu = [-u_\mu + (1 + ra_k)k_\mu]r^4\, d\Omega_4\, du \tag{5.4}$$

where $d\Omega_4 = d\vartheta_1\, d\vartheta_2\, d\vartheta_3\, d\phi \sin\vartheta_1 \sin^2\vartheta_2 \sin^3\vartheta_3$ is the element of solid angle in five dimensions. The angular integration can be handled via the relations

$$\int d\Omega_4 = \frac{8\pi^2}{3} \qquad \int d\Omega_4\, n^\alpha n^\beta = \frac{8\pi^2}{15}(\eta^{\alpha\beta} + u^\alpha u^\beta)$$
$$\int d\Omega_4\, n^\alpha n^\beta n^\gamma n^\kappa = \frac{8\pi^2}{105}[(\eta^{\alpha\beta} + u^\alpha u^\beta)(\eta^{\gamma\kappa} + u^\gamma u^\kappa) + (\eta^{\alpha\gamma} + u^\alpha u^\gamma)(\eta^{\beta\kappa} + u^\beta u^\kappa) + (\eta^{\alpha\kappa} + u^\alpha u^\kappa)(\eta^{\beta\gamma} + u^\beta u^\gamma)]. \tag{5.5}$$

The integral of the polynomial in odd powers of $n^\alpha := k^\alpha - u^\alpha$ vanishes.

We are now concerned with volume integration of (2.6). Although the surface element (5.4) contains the term which is proportional to r, the radiative part of the electromagnetic field six-momentum $p_{\rm rad}$ does not depend on the distance

$$p^\mu_{\rm rad} = \frac{e^2}{4\pi^2}\int_{-\infty}^{\tau} du\left(\frac{4}{5}u^\mu \dot{a}^2 - \frac{6}{35}a^2\dot{a}^\mu + \frac{3}{7}a^\mu (a^2)^{\cdot} + 2a^4 u^\mu\right). \tag{5.6}$$

(We denote by $(a^2)^{\cdot}$ the derivative da^2/du.) The reason is that $k_\alpha T^{\alpha\beta}_{\rm rad} = 0$. Since $k_\alpha T^{\alpha\beta}_{(-5)} = 0$, this term does not produce a change in radiation flux.

Volume integration of the bound part of the stress–energy tensor over the world tube Σ_r of constant radius r reveals that the bound energy–momentum is a function of the end points only:

$$p^\mu_{\rm bnd} = \frac{e^2}{4\pi^2}\left[\frac{3}{2}\frac{u^\mu(u)}{r^3} + \frac{12}{5}\frac{a^\mu(u)}{r^2} + 2\frac{a^2u^\mu(u)}{r}\right]_{u\to-\infty}^{u=\tau}. \tag{5.7}$$

(The fact is that the total (retarded) time derivatives arise from angular integration.) If the charged particle was asymptotically free in the remote past, we obtain the Coulomb-like self-energy of constant value. The upper limit drastically depends on the value of r. To evaluate the bound part of six-momentum in the neighbourhood of the particle we take the limit $r \to 0$ in (5.7). If r tends to zero, $p^\mu_{\rm bnd} \to \infty$. Hence equation (5.7) expresses the divergent part of the energy–momentum which is permanently attached to the charge.

6. Angular momentum of the retarded Liénard–Wiechert field in six dimensions

We now turn to the calculation of the angular momentum tensor (2.8). We calculate how much electromagnetic field angular momentum flows across a thin world tube enclosing a particle's trajectory (2.1) up to the observation time τ (see figure 1).

Decomposition of the angular momentum tensor density into the bound component and the radiative component is a very important for the calculation. Indeed, the former accounts for the angular momentum which remains bound to the charge while the latter corresponds to the amount of angular momentum which escapes to infinity.

We put (5.1) into (2.8) where the right-hand side of equation (3.2) should be substituted for y. It contains the term which is proportional to distance r. Vector surface element (5.4) also depends on r. In general, terms scaling as r^2 may appear. Since $T^{\alpha\beta}_{(-4)}$ is proportional to $k^\alpha k^\beta$ and the equality $k_\alpha T^{\alpha\beta}_{(-5)} = 0$ is fulfilled, the radiative component $M^{\mu\nu}_{\rm rad}$ does not depend on the distance:

$$M^{\mu\nu}_{\rm rad} = -\int_{-\infty}^{\tau} {\rm d}u \int {\rm d}\Omega_4\, r^4 u_\alpha \big(z^\mu T^{\alpha\nu}_{(-4)} - z^\nu T^{\alpha\mu}_{(-4)}\big)$$
$$- \int_{-\infty}^{\tau} {\rm d}u \int {\rm d}\Omega_4\, r^5 u_\alpha \big(k^\mu T^{\alpha\nu}_{(-5)} - k^\nu T^{\alpha\mu}_{(-5)}\big)$$
$$+ \int_{-\infty}^{\tau} {\rm d}u \int {\rm d}\Omega_4\, r^6 a_k k_\alpha \big(k^\mu T^{\alpha\nu}_{(-6)} - k^\nu T^{\alpha\mu}_{(-6)}\big). \qquad (6.1)$$

Having performed the angle integration we obtain

$$M^{\mu\nu}_{\rm rad} = \frac{e^2}{4\pi^2}\left\{\int_{-\infty}^{\tau} {\rm d}u \big(z^\mu P^\nu_{\rm rad} - z^\nu P^\mu_{\rm rad}\big)\right.$$
$$\left. + \int_{-\infty}^{\tau} {\rm d}u \left[\frac{4}{5}\,(a^\mu \dot{a}^\nu - a^\nu \dot{a}^\mu) + \frac{64}{35} a^2 (u^\mu a^\nu - u^\nu a^\mu)\right]\right\} \qquad (6.2)$$

where symbol $P_{\rm rad}$ denotes the integrand of equation (5.6).

The remaining terms involved in the angular momentum tensor density constitute the bound part of the angular momentum of the electromagnetic field:

$$M^{\mu\nu}_{\rm bnd} = \int_{\Sigma_r} {\rm d}\sigma_\alpha \left(z^\mu T^{\alpha\nu}_{\rm bnd} - z^\nu T^{\alpha\mu}_{\rm bnd}\right)$$
$$+ \int_{\Sigma_r} {\rm d}\sigma_\alpha \big[(y^\mu - z^\mu) T^{\alpha\nu}_{(-8)} - (y^\nu - z^\nu) T^{\alpha\mu}_{(-8)}\big]$$
$$+ \int_{\Sigma_r} {\rm d}\sigma_\alpha \big[(y^\mu - z^\mu) T^{\alpha\nu}_{(-7)} - (y^\nu - z^\nu) T^{\alpha\mu}_{(-7)}\big]$$
$$+ \int_{-\infty}^{\tau} {\rm d}u \int {\rm d}\Omega_4\, r^5 (-u_\alpha + k_\alpha) \big(k^\mu T^{\alpha\nu}_{(-6)} - k^\nu T^{\alpha\mu}_{(-6)}\big). \qquad (6.3)$$

Volume integration shows that the decomposition is meaningful. Indeed, the bound angular momentum depends on the state of motion of the particle at the end points only:

$$M^{\mu\nu}_{\rm bnd} = \frac{e^2}{4\pi^2}\left[z^\mu P^\nu_{\rm bnd} - z^\nu P^\mu_{\rm bnd} + \frac{12}{5}\frac{u^\mu a^\nu - u^\nu a^\mu}{r}\right]_{u\to-\infty}^{u=\tau}. \qquad (6.4)$$

By symbol $P_{\rm bnd}$ we mean the expression in between the squared brackets of equation (5.7).

It is worth noting that $M^{\mu\nu}_{\rm bnd}$ contains, apart from the usual term of type $z \wedge p_{\rm part}$, also an extra term which can be interpreted as the 'electromagnetic shadow' of the internal angular momentum of the 'bare' particle. It prompts that the bare 'core' possesses a 'spin'.

7. Energy–momentum and angular momentum balance equations

To derive the radiation reaction force in six dimensions we study the energy–momentum and angular momentum balance equations.

We calculate how much electromagnetic-field momentum and angular momentum flow across hypersurface Σ_r up to the proper time τ. We can do it at a time $\tau + \triangle\tau$. We demand that change in these quantities be balanced by a corresponding change in the quantities of the particle, so that the total energy–momentum and angular momentum are properly conserved.

Expressions (5.7) and (6.4) show that a charged particle cannot be separated from its bound electromagnetic 'cloud' which has its own energy–momentum and angular momentum. These quantities, together with their 'bare' mechanical counterparts, constitute the six-momentum and angular momentum of a 'dressed' charged particle. We proclaim the *finite* characteristics as those of true physical meaning.

It would not make sense to disrupt the bonds between different powers of small parameter r in (5.7). It is sufficient to assume that a charged particle possesses its own (already renormalized) six-momentum p_{part} which is transformed as a usual six-vector under the Poincaré group. The total energy–momentum of a closed system of an arbitrarily moving charge and its electromagnetic field is equal to the sum

$$P^{\mu} = p_{\mathrm{part}}^{\mu} + p_{\mathrm{rad}}^{\mu} \tag{7.1}$$

where p_{rad} is the radiative part (5.6) of the energy–momentum of the electromagnetic field which detaches itself from the charge and leads an independent existence.

With (6.4) in mind we assume that the already renormalized angular momentum tensor of the particle has the form

$$M_{\mathrm{part}}^{\mu\nu} = z^{\mu} p_{\mathrm{part}}^{\nu} - z^{\nu} p_{\mathrm{part}}^{\mu} + u^{\mu}\pi_{\mathrm{part}}^{\nu} - u^{\nu}\pi_{\mathrm{part}}^{\mu}. \tag{7.2}$$

In [9, 10, 15] the extra momentum π_{part} is due to additional degrees of freedom associated with acceleration involved in the Lagrangian function for a rigid particle.

The total angular momentum of our composite particle plus field system is written as

$$M^{\mu\nu} = M_{\mathrm{part}}^{\mu\nu} + M_{\mathrm{rad}}^{\mu\nu} \tag{7.3}$$

where M_{rad} is the radiative part (6.2) of the angular momentum of the electromagnetic field which depends on all previous motions of a source. Our next task is to derive expressions which explain how six-momentum and angular momentum of a charged particle depend on its velocity and acceleration, etc. By the differentiation of equation (7.1) we obtain the following energy–momentum balance equation:

$$\dot{p}_{\mathrm{part}}^{\mu} = -\frac{e^2}{4\pi^2}\left(\frac{4}{5}u^{\mu}\dot{a}^2 - \frac{6}{35}a^2\dot{a}^{\mu} + \frac{3}{7}a^{\mu}(a^2)^{\cdot} + 2a^4u^{\mu}\right). \tag{7.4}$$

(All the particle characteristics are evaluated at the time of observation τ.) Having differentiated (7.3) and taking into account (7.4) we arrive at the equality which does not contain $\dot{p}_{\mathrm{part}}$:

$$\begin{aligned} u^{\mu}\left(p_{\mathrm{part}}^{\nu} + \dot{\pi}_{\mathrm{part}}^{\nu}\right) - u^{\nu}\left(p_{\mathrm{part}}^{\mu} + \dot{\pi}_{\mathrm{part}}^{\mu}\right) + a^{\mu}\pi_{\mathrm{part}}^{\nu} - a^{\nu}\pi_{\mathrm{part}}^{\mu} \\ = -\frac{e^2}{4\pi^2}\left[\frac{4}{5}(a^{\mu}\dot{a}^{\nu} - a^{\nu}\dot{a}^{\mu}) + \frac{64}{35}a^2(u^{\mu}a^{\nu} - u^{\nu}a^{\mu})\right]. \end{aligned} \tag{7.5}$$

It is convenient to rewrite this system of fifteen linear equations in twelve variables $p_{\mathrm{part}}^{\alpha} + \dot{\pi}_{\mathrm{part}}^{\alpha}$ and $\pi_{\mathrm{part}}^{\alpha}$ as follows,

$$u \wedge (p_{\mathrm{part}} + \dot{\pi}_{\mathrm{part}}) + a \wedge \pi_{\mathrm{part}} = -\frac{e^2}{4\pi^2}\left[\frac{4}{5}a \wedge \dot{a} + \frac{64}{35}a^2 u \wedge a\right] \tag{7.6}$$

where the symbol $\wedge$ denotes the wedge product. Hence one has again

$$u \wedge \left(p_{\mathrm{part}} + \dot{\pi}_{\mathrm{part}} + \frac{e^2}{4\pi^2}\frac{64}{35}a^2 a\right) + a \wedge \left(\pi + \frac{e^2}{4\pi^2}\frac{4}{5}\dot{a}\right) = 0. \tag{7.7}$$

Their solutions involve three arbitrary scalar functions, e.g. M, μ and ν:

$$p_{\mathrm{part}}^{\beta} + \dot{\pi}_{\mathrm{part}}^{\beta} = Mu^{\beta} + \nu a^{\beta} - \frac{e^2}{4\pi^2}\frac{64}{35}a^2 a^{\beta} \tag{7.8}$$

$$\pi_{\mathrm{part}}^{\beta} = \mu a^{\beta} + \nu u^{\beta} - \frac{e^2}{4\pi^2}\frac{4}{5}\dot{a}^{\beta}. \tag{7.9}$$

Therefore, the rank of system (7.6) is equal to nine.

Scrupulous analysis of the consistency of (7.8) and (7.9) with six first-order differential equations (7.4) reveals that the six-momentum of a charged particle contains two (already renormalized) constants:

$$p^{\beta}_{\rm part} = mu^{\beta} + \mu\left(-\dot{a}^{\beta} + \frac{3}{2}a^2u^{\beta}\right) + \frac{e^2}{4\pi^2}\left[\frac{4}{5}\ddot{a}^{\beta} - \frac{8}{5}u^{\beta}(a^2)^{\cdot} - \frac{64}{35}a^2a^{\beta}\right] \tag{7.10}$$

(see the appendix). The first, m, looks like a rest mass of the charge. But the true rest mass is identical to the scalar product of the six-momentum and six-velocity [8]. Since the scalar product depends on the square of acceleration as well as its time derivative

$$m_0 = -(p_{\rm part}\cdot u) = m + \frac{\mu}{2}a^2 - \frac{e^2}{4\pi^2}\frac{2}{5}(a^2)^{\cdot} \tag{7.11}$$

the renormalization constant m is a formal parameter and its physical sense is not clear.

The second, μ, is intimately connected with the wedge product $u \wedge \pi_{\rm part} := s_{\rm part}$. With equation (7.2) in mind we call

$$s^{\alpha\beta}_{\rm part} = \mu(u^{\alpha}a^{\beta} - u^{\beta}a^{\alpha}) - \frac{e^2}{4\pi^2}\frac{4}{5}(u^{\alpha}\dot{a}^{\beta} - u^{\beta}\dot{a}^{\alpha}) \tag{7.12}$$

the internal angular momentum of the particle. But its magnitude is not constant

$$s^2 = -\frac{1}{2}s^{\rm part}_{\alpha\beta}s^{\alpha\beta}_{\rm part} = \mu^2a^2 + \mu\frac{e^2}{5\pi^2}(a^2)^{\cdot} + \frac{e^4}{25\pi^4}(\dot{a}^2 + a^4). \tag{7.13}$$

Therefore, this name cannot be understood literally.

Having substituted the right-hand side of equation (7.10) for the six-momentum of the particle in equation (7.4) we derive the Lorentz–Dirac equation of motion of a charged particle under the influence of its own electromagnetic field. The problem of including an external device requires careful consideration.

When considering the system under the influence of an external device the time derivative $\dot{P}$ of total momentum P is equal to external force $F_{\rm ext}$. It changes the energy–momentum balance equation (7.4) as follows:

$$\dot{p}^{\mu}_{\rm part} + \frac{e^2}{4\pi^2}\left(\frac{4}{5}u^{\mu}\dot{a}^2 - \frac{6}{35}a^2\dot{a}^{\mu} + \frac{3}{7}a^{\mu}(a^2)^{\cdot} + 2a^4u^{\mu}\right) = F^{\mu}_{\rm ext}. \tag{7.14}$$

The corresponding change of the total angular momentum $M^{\mu\nu}$ is defined by an external torque

$$\dot{M}^{\mu\nu} = z^{\mu}F^{\nu}_{\rm ext} - z^{\nu}F^{\mu}_{\rm ext}. \tag{7.15}$$

Expression (7.10) was first obtained by Kosyakov in [7, equation (37)]. The derivation is based upon consideration of energy–momentum conservation only. The author constructs an appropriate Schott term to ensure the orthogonality of the radiation reaction force to the particle six-velocity. Our derivation is based upon considerations of 21 conserved quantities corresponding to the symmetry of a closed system of charged particle and electromagnetic field under the Poincaré group. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction.

8. Conclusions

Our consideration is founded on the field and the interaction terms of action (1.1). They constitute the action functional which governs the propagation of the electromagnetic field produced by a moving charge (i.e., the Maxwell equations with point-like source).

We see that the particle part of initial action integral (1.1), which is proportional to the worldline length, is inconsistent with the others, $I_{\rm field}$ and $I_{\rm int}$. Indeed, the angular momentum tensor of a structureless particle

$$M^{\mu\nu}_{\rm part} = z^\mu p^\nu_{\rm part} - z^\nu p^\mu_{\rm part} \tag{8.1}$$

corresponds to $I_{\rm part}$ given by (1.2). Having analysed angular momentum balance equations one has again

$$u \wedge \left(p_{\rm part} + \frac{e^2}{4\pi^2}\frac{64}{35}a^2 a \right) + \frac{e^2}{4\pi^2}\frac{4}{5} a \wedge \dot{a} = 0 \tag{8.2}$$

instead of (7.7). Its solution is a motion with constant velocity, where $p^\mu_{\rm part}$ do not change. Hence the action functional based on the higher derivative Lagrangian for a 'rigid' relativistic particle [9–11, 15] should be substituted for $I_{\rm particle}$ in (1.2). It is sufficient to renormalize *all* the divergences arising in six-dimensional electrodynamics (these connected with bound six-momentum and those associated with bound angular momentum of the electromagnetic field). The variation of modified action with respect to particle variables results in the appropriate equation of motion of a charged particle in response to the electromagnetic field.

A surprising feature of the study of Poincaré invariance of the dynamics of a closed particle plus field system is that the conservation laws determine the form of individual characteristics of the particle such as the momentum and the angular momentum. The fact is that a charged particle cannot be separated from its bound electromagnetic 'cloud' which has its own momentum and angular momentum. These quantities together with the corresponding characteristics of the bare 'core' constitute the momentum and angular momentum of a 'dressed' charged particle.

So, in four dimensions the momentum of a bare 'core' is proportional to its four-velocity. An electromagnetic 'cloud' renormalizes the bare mass and adds the term which is proportional to the four-acceleration (see equation (1.6)). The extra term can be obtained from the angular momentum balance equation [14]. In six dimensions a bare charge should possess (non-conventional) internal angular momentum

$$s_0^{\mu\nu} = \mu_0 \left(u^\mu a^\nu - u^\nu a^\mu \right) \tag{8.3}$$

with a magnitude which is proportional to the square of the acceleration (see equation (7.13)). Its six-momentum is not proportional to the six-velocity

$$p_0^\mu = m_0 u^\mu + \mu_0 \left(-\dot{a}^\mu + \tfrac{3}{2} a^2 u^\mu \right). \tag{8.4}$$

The energy–momentum and angular momentum balance equations give the six-momentum (7.10) of a 'dressed' charged particle which coincides with that obtained in [7].

It is interesting to consider the motion of test particles (i.e., point charges which themselves do not influence the field). In four dimensions the limits $e \to 0$ and $m \to 0$ with their ratio being fixed result in the Maxwell–Lorentz theory of a test particles. The momentum of a test particle is proportional to its four-velocity, the loss of energy due to radiation is too small to be observed. In six dimensions the test particle is the *rigid* particle. Its momentum is not parallel to the six-velocity. The problem of motion of such particles in an external electromagnetic field is considered in [18].

One way of continuing the present work would be the investigation of arbitrary even dimensions. It is sufficient to limit our calculations to the radiative parts of energy–momentum and angular momentum carried by an electromagnetic field. The reason is that they determine the energy–momentum and angular momentum balance equations which allow us to establish the radiation reaction force.

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Appendix

Since $(u \cdot a) = 0$, the scalar product of particle six-velocity on the first-order time derivative of particle six-momentum (7.4) is as follows:

$$(\dot{p}_{\rm part} \cdot u) = \frac{e^2}{4\pi^2}\left(\frac{4}{5}\dot{a}^2 + \frac{64}{35}a^4\right). \tag{A.1}$$

Similarly, the scalar product of particle acceleration on the particle six-momentum (7.8) is given by

$$(p_{\rm part} \cdot a) = \nu a^2 - \frac{e^2}{4\pi^2}\frac{64}{35}a^4 - (\dot{\pi}_{\rm part} \cdot a). \tag{A.2}$$

Summing up (A.1) and (A.2) we obtain

$$(p_{\rm part} \cdot u)^{\cdot} = \nu a^2 + \frac{e^2}{4\pi^2}\frac{4}{5}\dot{a}^2 - (\dot{\pi}_{\rm part} \cdot a). \tag{A.3}$$

On the other hand, the time derivative $(p_{\rm part} \cdot u)^{\cdot}$ of the scalar product of particle velocity on the momentum (7.8) is written as

$$(p_{\rm part} \cdot u)^{\cdot} = -\dot{M} - (\dot{\pi}_{\rm part} \cdot u)^{\cdot}. \tag{A.4}$$

Subtracting (A.4) from (A.3), one has again

$$\dot{M} = -\nu a^2 - \frac{e^2}{4\pi^2}\frac{4}{5}\dot{a}^2 - (\ddot{\pi}_{\rm part} \cdot u). \tag{A.5}$$

Further we calculate the scalar product of the second-order derivative of (7.9) on the velocity of the particle

$$(\ddot{\pi}_{\rm part} \cdot u) = -2\dot{\mu}a^2 - \frac{3}{2}\mu(a^2)^{\cdot} - \ddot{\nu} - \nu a^2 + \frac{e^2}{4\pi^2}\left(\frac{8}{5}(a^2)^{\cdot\cdot} - \frac{4}{5}\dot{a}^2\right). \tag{A.6}$$

Having substituted it into the previous equation we arrive at the following differential equation:

$$\dot{M} = 2\dot{\mu}a^2 + \frac{3}{2}\mu(a^2)^{\cdot} + \ddot{\nu} - \frac{e^2}{4\pi^2}\frac{8}{5}(a^2)^{\cdot\cdot}. \tag{A.7}$$

It can be solved iff the scalar μ does not change with time

$$M = m + \frac{3}{2}\mu a^2 + \dot{\nu} - \frac{e^2}{4\pi^2}\frac{8}{5}(a^2)^{\cdot}. \tag{A.8}$$

Having substituted it into (7.8) and taking into account the time derivative of (7.9), we derive expression (7.10) for the components of the six-momentum of the charged particle. It depends on two renormalization constants, m and μ.

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